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BARGAINING SET AND KERNEL OF BIG BOSS GAMES

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Abstract

The main objective of the paper is to describe the bargaining set and the kernel of big boss games and information market games. The kernel turns out to consist of one point, the τ-value of the game, while the bargaining set coincides with the core of the game. Furthermore, for a special subclass of big boss games a stable set is given.

Keywords: Cooperative game, bargaining set, kernel.

1 Introduction

In a recent paper Muto et al. [6] defined a class of cooperative games intended to model the trading of information between one informed firm and other initially non-informed firms. For those cooperative games—information market games—the structure of the core is extremely simple, the nucleolus coincides with the τ-value and the core is a stable set if and only if the game is convex. Later on, it was realized that all of these properties are shared by a larger class of cooperative games— big boss games. Muto et al. [7] gives the definition of a big boss game and proves that most results of Muto et al. [6] also hold for big boss games.

The purpose of this paper is to find the bargaining set and the kernel of big boss games. For a special class of big boss games we also give a stable set.

In order to make the paper as self-contained as possible we repeat the most crucial definitions and the more important results of [6] and [7].

A cooperative game (N, v) is called an *information market game* if there is a player $i_0 \in N$ and non-negative real numbers $\{r_T\}_{T\subset N, T\neq\emptyset}$ such that for each coalition $S \subset N$

$$v(S) = \begin{cases} \sum_{S\cap T\neq\emptyset} r_T & \text{if } i_0 \in S \\ 0 & \text{if } i_0 \notin S \end{cases}$$

We refer to Muto et al. [6] for the economical background of these games and the proof of the following properties.

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- The core $\mathcal{C}(v) = \{x \in \mathbf{R}^N_+ \,|\, 0 \leq x_i \leq M_v(i) = r_{\{i\}} \text{ for all } i \in N\}$.
- The nucleolus and the τ-value are the same: $n(v)_i = \tau(v)_i = \frac{1}{2}M_v(i)$ for all $i \neq i_0$.
- The game is convex if and only if the core is a stable set.

A more comprehensive class of games has been introduced in Muto [7].

A cooperative game (N, v) is a *big boss game* if

1. The game is monotonic i.e. $v(S) \leq v(T)$ if $S \subset T$.
2. There is a player $i_0 \in N$ such that $v(S) = 0$ if $i_0 \notin S$.
3. If $i_0 \in S$, then $v(N) - v(S) \geq \sum_{i \notin S} M_v(i)$ where $M_v(i) = v(N) - v(N \setminus \{i\})$.

It is rather easy to see that information market games are examples of big boss games and in Muto et al. [7] it is proved that the above mentioned properties also hold for big boss games. Even if we replace the first condition for a big boss game by the weaker condition:

1*. $v \geq 0$ and $M_v(i) \geq 0$ for all $i \in N$.

the properties remain true with the only exception that the third property should be read as:

- The game is convex if and only if the game is monotonic and the core is a stable set.

In this paper we will call games satisfying the conditions $(1^*), (2)$ and (3) big boss games. For the games satisfying the original definition of Muto et al. [7] we reserve the term *monotonic big boss game*.

In the sequel we use the following notations:

The *core* of a cooperative game (N, v) will be denoted by $\mathcal{C}(v)$.

The expressions $\mathcal{M}(v)$ and $\mathcal{K}(v)$ we reserve for the *bargaining set* and the *kernel* of the game (N, v).

The set $\{x \in \mathbf{R}^N \,|\, x_i \geq v(\{i\}) \text{ for all } i \in N \text{ and } \sum_{i \in N} x_i = v(N)\}$—the set of imputations of the game (N, v)—will be denoted by $\mathcal{A}(v)$.

2 The bargaining set of big boss games

Let $x \in \mathcal{A}(v)$ for a cooperative game (N, v). An *objection* of a player i against a player j (with respect to the imputation x) consists of a coalition S with the property $i \in S \subset N \setminus \{j\}$ and a vector $y \in \mathbf{R}^S$ such that $y_i > x_i$ for all $i \in S$ and $y(S) := \sum_{i \in S} y_i \leq v(S)$. A *counter objection* of player j against player i (with respect to the imputation x and the objection (S, y)) consists of a coalition T with $j \in T \subset N \setminus \{i\}$ and a vector $z \in \mathbf{R}^T$ such that $z_i \geq y_i$ for all $i \in S \cap T$, $z_j \geq x_j$ for all $j \in T \setminus S$ and $z(T) \leq v(T)$. An imputation $x \in \mathcal{A}(v)$ is an element of the *bargaining set* $\mathcal{M}(v)$ of the game (N, v) if every objection of any player against any other player meets a counter objection. This concept, introduced by Aumann and Maschler [1], was intended to provide a solution for games with an empty core. As soon as the game has imputations, the bargaining set is well-defined and non-empty. It would have been nice if the bargaining set coincides with the core in the case both are non-empty. In general, however, this is not the case. But, in Maschler, Peleg and Shapley [4], it has been proved that for convex games the core and the bargaining set coincide. The next theorem gives an other class of games (including the class of big boss games) exhibiting the same property:[3]

Theorem 1 *If (N, v) is a cooperative game with the properties:*

1. *there is a player $i_0 \in N$ such that $v(S) = 0$ if $i_0 \notin S$*
2. *for all coalitions $S \subset N$ we have $0 \leq v(S) \leq v(N)$*

then $\mathcal{M}(v) = \mathcal{C}(v)$.

Proof: [Maschler, private communication] If $x \in \mathcal{C}(v)$, then no objection is possible (since $x(S) \geq v(S)$ for all $S \subset N$) and therefore, $x \in \mathcal{M}(v)$. Suppose, conversely, that $x \in \mathcal{A}(v) \setminus \mathcal{C}(v)$ Then, there is a coalition $S \subset N$ such that $x(S) < v(S)$. Because $x(N) = v(N) \geq v(S)$, there is a player $j \in N \setminus S$ with $x_j > 0$. Then (S, y) with $y_i = x_i + \varepsilon$ for all $i \in S$ and $|S|\varepsilon = v(S) - x(S)$, is an objection of player i_0 against player j. Note that $v(S) > 0$ and $i_0 \in S$, by consequence. There is no counter objection of player j against player i_0 possible, since $j \in T \subset N \setminus \{i_0\}$ implies that $v(T) = 0$. To be a counter objection, there should be a vector $z \in \mathbf{R}^T$ with $z_i \geq x_i \geq 0$ for all $i \in T$ and $z_j \geq x_j > 0$. Then $z(T) > 0 = v(T)$. Hence, $x \notin \mathcal{M}(v)$. □

Corollary: *For big boss games the core and the bargaining set are the same.* □

[3]The history of this theorem is a little curious. Initially the theorem has been proved by the authors for big boss games only. Then, in a letter, Maschler provided a proof of the theorem for *monotonic* games satisfying condition 1 of the theorem. But in fact, only condition 1 and 2 were actually used in the proof. The proof we give here is essentially Maschlers'.

3 The kernel of big boss games

Let us repeat the definition of the kernel. Suppose, (N,v) is a cooperative game with $\mathcal{A}(v) \neq \emptyset$ and $x \in \mathcal{A}(v)$. For every pair (i,j) of players we define:

$$s_{i,j}(x) = \max_{S: i \in S \subset N \setminus \{j\}} v(S) - x(S).$$

The kernel $\mathcal{K}(v)$ consists of the imputations x with the property that for each pair (i,j) with $x_j > v(\{j\})$

$$s_{i,j}(x) \leq s_{j,i}(x)$$

. The kernel was introduced in Davis and Maschler [2]. It has been shown to contain the nucleolus (Schmeidler [8]) and to be a subset of the bargaining set ([2]). For big boss games the nucleolus(= the τ-value) is the only element of the kernel.

Theorem 2 *The kernel of a big boss game consists of one point x with:*

$$x_i = \frac{1}{2} M_v(i) \text{ for all } i \neq i_0$$

Proof: Let x be an element of the kernel $\mathcal{K}(v)$. First we prove that $x_i \leq \frac{1}{2}M_v(i)$ for all $i \neq i_0$. We may assume that $x_i > 0 = v(\{i\})$. Then $s_{i_0,i} \leq s_{i,i_0}$. This means that for all coalitions S with $i_0 \in S \subset N \setminus \{i\}$

$$v(S) - x(S) \leq \max_{i \in T \subset N \setminus \{i_0\}} v(T) - x(T).$$

Because $i_0 \notin T$, $v(T) = 0$ and $s_{i,i_0}(x) = \max_{i \in T \subset N \setminus \{i_0\}} -x(T) = -x_i$. If we take $S = N \setminus \{i\}$ then we find $v(N \setminus \{i\}) - x(N \setminus \{i\}) \leq -x_i$. Because $x(N \setminus \{i\}) = x(N) - x_i = v(N) - x_i$, we get $v(N \setminus \{i\}) - v(N) + x_i \leq -x_i$ or $x_i \leq \frac{1}{2}M_v(i)$ for all $i \neq i_0$. If $M_v(i) = 0$ for all $i \neq i_0$, then $x_i = 0$ for all $i \neq i_0$ and we are done. In all other cases

$$v(N) - v(\{i_0\}) > \frac{1}{2}\sum_{i \neq i_0} M_v(i) \geq \sum_{i \neq i_0} x_i$$

by the third condition for a big boss game and $x_{i_0} > v(\{i_0\})$. Then $s_{i,i_0}(x) = -x_i \leq v(T) - x(T)$ for at least one coalition T with $i_0 \in T \subset N \setminus \{i\}$. Since $i_0 \in T$, we have

$$v(T) \leq v(N) - \sum_{j \notin T} M_v(j) = v(N \setminus \{i\}) - \sum_{j \neq i, j \notin T} M_v(j) \leq v(N \setminus \{i\}) - \sum_{j \in N \setminus (T \cup \{i\})} x_j$$

Substracting $x(T)$ from both sides, we find $v(T) - x(T) \leq v(N \setminus \{i\}) - x(N \setminus \{i\})$. This means that we can take $T = N \setminus \{i\}$. Hence $-x_i \leq v(N/\{i\}) - x(N \setminus \{i\})$ and $2x_i \geq M_v(i)$. For all elements of the kernel $x_i = \frac{1}{2}M_v(i)$ for all $i \neq i_0$. Then $x_{i_0} = v(N) - \frac{1}{2}M_v(N \setminus \{i_0\})$. □

Remark: In Muto et al. [7] we found an explicit formula for the nucleolus of a big boss game by using a criterion of Kohlberg [3]. Indirectly we rediscover this result by combining the result of the forgoing theorem with the fact that the kernel always contains the nucleolus (Schmeidler [8]).

4 Stable sets for a subclass of big boss games

Stable sets are defined in terms of domination. For a game (N, v) let $x, y \in \mathcal{A}(v)$ and $S \subset N$. We say that x *dominates* y via S (notation $x \, dom_S \, y$) if $x_i > y_i$ for all $i \in S$ and $x(S) \leq v(S)$ and that x dominates y (notation $x \, dom \, y$) if there is a coalition S such that $x \, dom_S \, y$. A subset $K \subset \mathcal{A}(v)$ is called a *stable set* for (N, v) if

1. If $x, y \in K$,then x does not dominate y (internal stability).
2. For each $y \in \mathcal{A}(v) \setminus K$ there is an element x of K such that $x \, dom \, y$ (external stability).

The study of stable sets of information market games and big boss games is still an open area. Only in the paper [5] stable sets are described for certain 4-person information market games. In this section we will describe a stable set for big boss games satisfying the (rather strong) conditions (S.1),(S.2) and (S.3) below. Consider cooperative games (N, v) with

(S.1) v is monotonic

(S.2) there is a player $i_0 \in N$ such that $v(S) = 0$ if $i_0 \notin S$

(S.3) there is a non-negative real number r such that for all coalitions S with $S \underset{\neq}{\supset} \{i_0\}$, $v(S) = \sum_{i \in S \setminus \{i_0\}} M_v(i) + v(\{i_0\}) + r$.

Remark 1: Information market games with these properties are of the following type: there is a market accessible for all initially non-informed firms $r = r_{N \setminus \{i_0\}}$, there are markets accessible for i_0: $v(\{i_0\}) = \sum_{i_0 \in T} r_T$ and each initially non-informed firm i has a market only accessible for himself: $M_v(i) = r_{\{i\}}$. Summarizing we can say: $r_T = 0$ if $2 \leq |T| \leq n-2$ and $i_0 \notin T$.

Remark 2: Cooperative games satisfying the conditions (S.1), (S.2) and (S.3) are big boss games. Only the third property has to be checked. For $i_0 \in S$ and $|S| \geq 2$ we have

$$v(N) - v(S) = \sum_{i \in N \setminus \{i_0\}} M_v(i) - \sum_{i \in S \setminus \{i_0\}} M_v(i) = \sum_{i \in N \setminus S} M_v(i)$$

For $S = \{i_0\}$, then

$$v(N) - v(\{i_0\}) = \sum_{i \in N \setminus \{i_0\}} M_v(i) + r$$

For games of this kind we can give a stable set explicitly.

Theorem 3 *For cooperative games (N,v) satisfying (S 1), (S.2) and (S.3) the set $K = \mathcal{C}(v) \cup K_1$ is a stable set where*

$$K_1 = \{x \in \mathcal{A}(v) \mid x_i = M_v(i) + t \text{ for all } i \neq i_0 \text{ and } 0 \leq t \leq \frac{r}{n-1}\}$$

Proof: (A) [internal stability] Suppose $x, y \in K$ and $x\, dom_S\, y$. Then y is no element of $\mathcal{C}(v)$ (core elements are undominated), $i_0 \in S$ (because $0 \leq y(S) < x(S) \leq v(S)$) and $S \neq \{i_0\}$ (because $x_{i_0} > y_{i_0} \geq v(\{i_0\})$). Since $y \in K \setminus \mathcal{C}(v)$, we have $y_i > M_v(i)$ for all $i \neq i_0$. Hence an element of K dominating y, lies in K_1 too. Then $x_i > y_i$ not only for $i \in S \setminus \{i_0\}$ but for all $i \neq i_0$. This means that $x(N \setminus \{i_0\}) > y(N \setminus \{i_0\})$ and, hence, $x_{i_0} < y_{i_0}$. There is no domination inside K: the internal stability has been proved.
(B) [external stability] Let $y \in \mathcal{A}(v) \setminus K$. We have to prove that y is dominated by a core element or by an element of K_1.

(i) Let us suppose that $y_i < M_v(i)$ for at least one $i \neq i_0$. Let $S = \{i \in N \mid i = i_0 \text{ or } y_i < M_v(i)\}$. Suppose, $y(S) \geq v(S) = \sum_{i \in S \setminus \{i_0\}} M_v(i) + v(\{i_0\}) + r$ (by (S.3)). Further, $y(N \setminus S) \geq M_v(N \setminus S)$ (by the definition of S). Hence, $y(N) \geq \sum_{i \in N \setminus \{i_0\}} M_v(i) + v(\{i_0\}) + r = v(N)$ and $y(N \setminus S) = M_v(N \setminus S)$. This means that $y \in \mathcal{C}(v) \subset K$ in contradiction with the choice of y. Choose $\varepsilon > 0$ such that $|S|\varepsilon = y(N \setminus S) - M_v(N \setminus S) > 0$ and define $x_i = \min(y_i + \varepsilon; M_v(i))$ for all $i \in S \setminus \{i_0\}$, $x_j = M_v(j)$ if $j \notin S$ and $x_{i_0} = v(N) - x(N \setminus \{i_0\})$. It is clear that $x\, dom_S\, y$ and $x \in \mathcal{C}(v)$.

(ii) Now, let us suppose that $y_i \geq M_v(i)$ for all $i \neq i_0$. Let $i_1 \neq i_0$ be a player with $y_{i_1} - M_v(i_1) \leq y_i - M_v(i)$ for all $i \neq i_0$. Take $\varepsilon = y(N \setminus \{i_0\}) - M_v(N \setminus \{i_0\}) - (n-1)(y_{i_1} - M_v(i_1))$. Then $\varepsilon > 0$ because $\varepsilon = 0$ would mean that $y \in K_1$. We prove that the point $x \in K_1 \subset \mathcal{A}(v)$ with $x_i - M_v(i) = y_{i_1} - M_v(i_1) + n^{-1}\varepsilon$ for all $i \neq i_0$ dominates y via $\{i_0, i_1\}$. First, $x(N \setminus \{i_0\}) = M_v(N \setminus \{i_0\}) + (n-1)(y_{i_1} - M_v(i_1) + n^{-1}\varepsilon) = y(N \setminus \{i_0\}) - n^{-1}\varepsilon$. From the efficiency follows, $x_{i_0} = y_{i_0} + n^{-1}\varepsilon$. Hence we find $x_{i_0} > y_{i_0}$ and $x_{i_1} > y_{i_1}$. Finally we prove that

$$\begin{aligned} x_{i_0} + x_{i_1} = v(N) - x(N \setminus \{i_0, i_1\}) &= M_v(N \setminus \{i_0\}) + v(\{i_0\}) + \\ r - M_v(N \setminus \{i_0, i_1\}) &- (n-2)(y_{i_1} - M_v(i_1) + n^{-1}\varepsilon) = \\ M_v(i_1) + v(\{i_0\}) &+ r - (n-2)(y_{i_1} - M_v(i_1) + n^{-1}\varepsilon) \end{aligned}$$

From (S.3) we have $v(\{i_0, i_1\}) = M_v(i_1) + v(\{i_0\}) + r$. Therefore, we have $x\, dom_{\{i_0,i_1\}}\, y$ and $x \in K_1$. □

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